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A Result on Best Proximity Pair of Two Sets

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Let X be a normed linear space and F, G nonvoid subsets of X. If $f_0 \in F$ and $g_0 \in G$ satisfy

$$\|f_0 - g_0\| = d(F, G) \stackrel{\text{def}}{=} \inf\{\|f - g\| : f \in F,\ g \in G\},$$

then we call (f_0, g_0) a best proximity pair of F and G.

Several authors have studied such proximity pairs. In [1, p. 385] I. Singer claims the following

THEOREM 1. *Let X be a metric space endowed with the metric ρ and F, G nonvoid, boundedly compact closed sets in X. Then there exist elements $f_0 \in F$ and $g_0 \in G$ such that*

$$\rho(f_0, g_0) = \rho(F, G) \stackrel{\text{def}}{=} \inf(\rho(f, g) : f \in F,\ g \in G\}.$$

Theorem 1 is false, as shown by the following example. Let E^2 be the Euclidean 2-space and let

$$F = \{(x, y) \in E^2 : x \geqslant 0,\ y \leqslant 0\},\ G = \left\{(x, y) \in E^2 : x > 0,\ y \geqslant \frac{1}{x}\right\}.$$

Then F and G are nonvoid boundedly compact sets. But there do not exist $f_0 \in F$, $g_0 \in G$ satisfying

$$\rho(f_0, g_0) = \rho(F, G)$$

because

$$F \cap G = \varnothing, \qquad \rho(F, G) = 0.$$

It is easy to see that Theorem 1 is true if we add the condition "F or G is bounded."

In 1974, D. V. Pai [2] gave the following theorem:

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THEOREM 2. *Let X be a uniformly convex Banach space and F, G two closed convex subsets, and let one of them be compact. Then there exist $f_0 \in F$ and $g_0 \in G$ such that*

$$\|f_0 - g_0\| = d(F, G).$$

In 1980, B. N. Sahney and S. P. Singh [3] gave the following theorem in a strictly convex Banach space.

THEOREM 3. *Let X be a strictly convex Banach space and F a closed, convex, locally compact subset of X, and let G be a compact, convex subset of X. Then there exist $f_0 \in F$ and $g_0 \in G$ such that*

$$\|f_0 - g_0\| = d(F, G).$$

Unfortunately, the proof of Theorem 3 in [3] is incorrect. However, we can easily prove Theorem 3 without the condition "X is strictly convex." We extract two sequences $\{f_n\} \subset F$ and $\{g_n\} \subset G$ such that $\lim_{n \to \infty} \|f_n - g_n\| = d(F, G)$. Since G is compact and F is closed, locally compact, we can extract from $\{f_n\}$ and $\{g_n\}$ two susequences $\{f_{n_k}\}$ and $\{g_{n_k}\}$ such that $\lim_{k \to \infty} f_{n_k} = f_0 \in F$ and $\lim_{k \to \infty} g_{n_k} = g_0 \in G$. We have then

$$\|f_0 - g_0\| = \lim_{k \to \infty} \|f_{n_k} - g_{n_k}\| = d(F, G),$$

which completes the proof.

In order to improve Theorems 1–3, we introduce several concepts.

Let F and G be subsets of X, F is said to be *weakly sequentially compact* if for each sequence in F, there always exists a subsequence weakly converging to a point of F. G is said to be *proximinal with respect to F* if for each $f \in F$, there exists a best approximation of f in G. The concept of proximinality can be easily extended to a metric space.

Obviously, a compact set is a weakly sequentially compact set. A boundedly compact set in metric space and a closed convex set in a uniformly convex Banach space are all proximinal with respect to the total space.

Our main result in this paper is the following theorem, which is an improvement of Theorems 1–3.

THEOREM 4. *Let F and G be two nonvoid sets of a normed linear space X. If F is weakly sequentially compact and G is convex and proximinal with respect to F, then a best proximity pair of F and G exists. For a metric space X, we assume instead that F is compact and G is proximinal with respect to F.*

Proof. We extract $\{f_n\} \subset F$ and $\{g_n\} \subset G$ such that

$$\lim_{n \to \infty} \|f_n - g_n\| = d(F, G).$$

Since F is weakly sequentially compact, without loss of generality, we can assume

$$W\text{-}\lim_{n\to\infty} f_n = f_0 \in F.$$

Since for all $m>0$, $W\text{-}\lim_{n\to\infty, n>m} f_n = f_0$, we have $f_0 \in \overline{\mathrm{CO}}\{f_n : n>m\}$. Therefore, if we let $m_0=0$, we have then, for $k=1, 2, 3, \dots$, a natural number $m_k (m_k \uparrow)$ and numbers $\alpha_j^{(k)}>0$ $(j=m_{k-1}+1, \dots, m_k)$, $\sum_{j=m_{k-1}+1}^{m_k} \alpha_j^{(k)}=1$, such that

$$\left\| \sum_{j=m_{k-1}+1}^{m_k} \alpha_j^{(k)} f_j - f_0 \right\| < \frac{1}{k}.$$

Let

$$u_k = \sum_{j=m_{k-1}+1}^{m_k} \alpha_j^{(k)} f_j, \qquad v_k = \sum_{j=m_{k-1}+1}^{m_k} \alpha_j^{(k)} g_j, \qquad k=1, 2, \dots,$$

then

$$\lim_{k\to\infty} u_k = f_0, \qquad v_k \in G, k=1, 2, \dots$$

and

$$\|u_k - v_k\| \leqslant \sum_{j=m_{k-1}+1}^{m_k} \alpha_j^{(k)} \|f_j - g_j\| \leqslant \max_{m_{k-1}+1 \leqslant j \leqslant m_k} \|f_j - g_j\|.$$

Since $u_k \to f_0$, $\|f_k - g_k\| \to d(F, G))$ $(k\to\infty)$, for any $\varepsilon>0$, there exists $K>0$ such that if $k \geqslant K$, then we have

$$\|u_k - f_0\| < \varepsilon, \qquad \|f_k - g_k\| < d(F, G) + \varepsilon.$$

Hence for $k \geqslant K$ and $m_{k-1} \geqslant K$ we have

$$\|u_k - v_k\| < d(F, G) + \varepsilon.$$

Since G is proximinal with respect to F, we have $g_0 \in G$ such that $\|f_0 - g_0\| = \inf_{g\in G} \|f_0 - g\|$. Then

$$d(F, G) \leqslant \|f_0 - g_0\| \leqslant \|f_0 - v_k\| \qquad (k=1, 2, \dots).$$

Therefore we have, for $k \geqslant K$ and $m_{k-1} \geqslant K$

$$\begin{aligned} d(F, G) &\leqslant \|f_0 - g_0\| \leqslant \|f_0 - v_k\| \\ &\leqslant \|f_0 - u_k\| + \|u_k - v_k\| \\ &< d(F, G) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we have $\|f_0 - g_0\| = d(F, G)$, i.e., (f_0, g_0) is one of the best proximity pairs of F and G.

If X is a metric space, F compact and G proximinal with respect to F, the proof is similar to the former proof; we omit it. The proof is completed.

From the proof of Theorem 4, we can get

COROLLARY 1. *Let X be a normed linear space and F, G two nonvoid subsets of X. If either F is compact and G is proximinal with respect to F or F is locally compact and G is bounded and proximinal with respect to F, then a best proximity pair of F and G exists.*

COROLLARY 2. *Let X be a reflexive Banach space and F, G two nonvoid subsets of X. If F is bounded, weakly closed, and G is closed convex, then a best proximity pair of F and G exists.*

Indeed, any bounded, weakly closed subset of a reflexive Banach space is weakly sequentially compact. Since any closed convex subset of this space is proximinal with respect to the total space, this corollary can be obtained from Theorem 4.

The characterizations and uniqueness of best proximity pairs have been discussed in [4].

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